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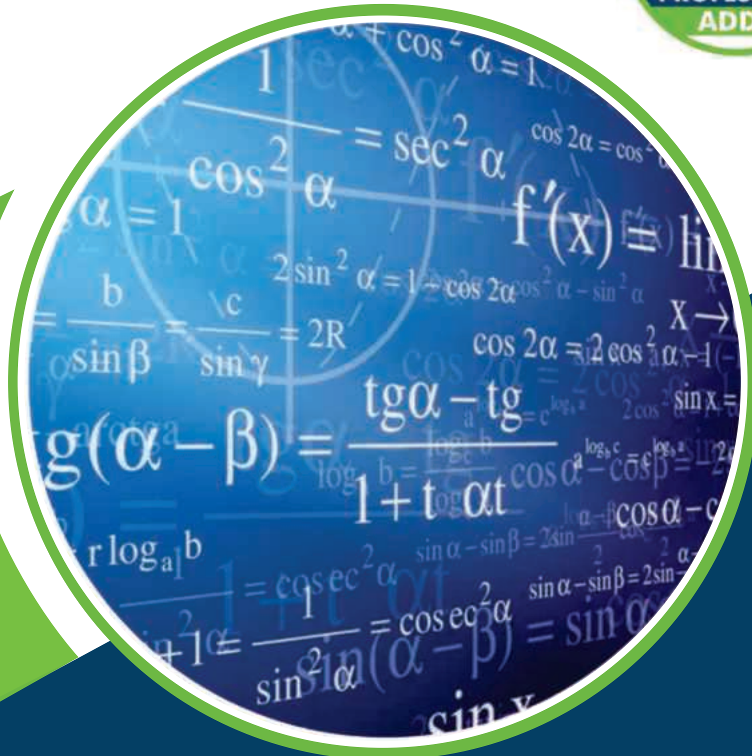

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# UGC NET

## MATHEMATICS PAPER II NOTES

A COMPLETE PREPARATION FOR UGC NET MATHEMATICS

- Latest Syllabus
- Unit-wise MCQ
- Well Researched Content

# CSIR UGC NET Part A

# What To Study

## Numerical Ability

This section tests your fundamental math and logic skills.[1] Focus on understanding the concepts rather than memorizing complex formulas.

- **Number and Simplification**:
  - **What to Study**: Master the **BODMAS** rule (Brackets, Orders, Division, Multiplication, Addition, Subtraction).Practice problems with fractions, decimals, and basic arithmetic operations.
- **Time, Speed, and Distance**:
  - **What to Study**: The core formula Distance=Speed×Time.[3] Focus on questions involving average speed, relative speed (**Example** two trains crossing each other), and boats/streams.
- **Time and Work**:
  - **What to Study**: Problems where you calculate the time taken by individuals or groups to complete a task. Also, cover "Pipes and Cisterns" problems, which use the same logic.
- **Percentage, Profit, and Loss**:
  - **What to Study**: This is a **high-priority** area. Understand how to calculate percentage increase/decrease, successive discounts, and

# How to study

## Step 1: Start with a Diagnostic Test
Before you begin studying, take one **full Previous Year's Question Paper (PYP)** and attempt Part A under timed conditions (around 30 minutes).

- **Purpose**: This isn't to test your knowledge, but to help you understand the exam's reality.
- **What to Analyze**:
  - Which topics did you find easiest?
  - Which questions took the most time?
  - What was the nature of the questions – were they calculation-heavy or logic-based?
- **Outcome**: You'll get a clear picture of your strengths, weaknesses, and the exam pattern.

## Step 2: Prioritize and Select Your Topics
You only need to correctly answer **15 out of 20 questions**. You don't need to master every topic.

- **Create a Target List**: Based on your diagnostic test and PYP analysis, select 8-10 topics. This list should be a mix of:
  1. **High-Frequency Topics**: Topics that appear every year (**Example** Series Formation, Data Interpretation, Geometry, Time

# Exam Tips

## Tip 1: The First 5 Minutes are Golden

Do not start solving the first question immediately. Instead, use the first 3-5 minutes to **scan all 20 questions** in Part A.

- **What to look for**: Quickly identify the questions that seem easiest and most familiar to you. Mentally bookmark them. Questions with diagrams (geometry, graphs), short questions, and direct series or ratio problems are often good candidates.
- **The Goal**: Create a mental roadmap of the **15 questions** you will attempt. This strategy prevents you from getting stuck on difficult questions at the beginning and ensures you don't miss easy ones at the end.

## Tip 2: Master the Art of Question Selection

Your primary task is not to solve questions in order, but to find and solve the easiest 15.

- **Categorize Questions**: As you solve, mentally put each question into one of three categories:
  1. **Easy**: You know how to solve it instantly. Do these first.
  2. **Medium**: You think you can solve it, but it might take some time. Mark these for the second round.
  3. **Hard**: You have no idea how to approach it or it looks very time-consuming. **Skip these completely.**

# CSIR UGC NET Part A

## Cracking CSIR-UGC NET Part A

A Data-Driven Analysis for Aspiring Researchers

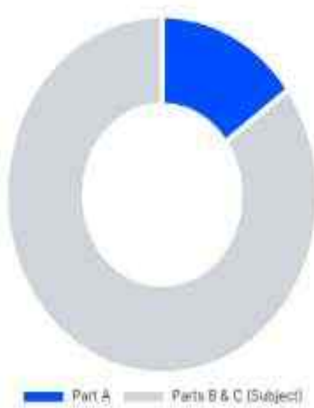
### The Big Picture: Market Overview

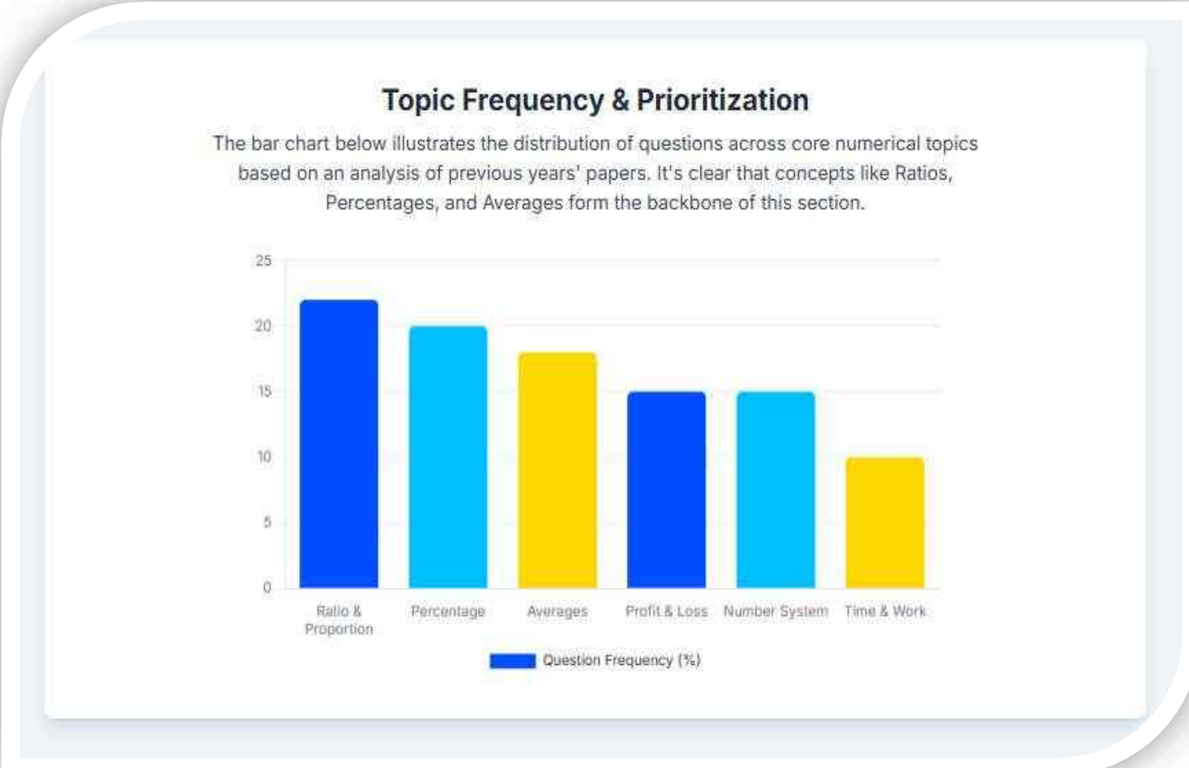
Part A represents a critical segment of the CSIR-UGC NET examination. While subject knowledge is paramount, this general aptitude section holds the key to achieving a competitive edge. It consistently contributes 30 marks, acting as a significant rank differentiator. Understanding its structure is the first step in market penetration.

| 20 | 15 | 30 | -0.5 |
|---|---|---|---|
| Total Questions | Max Attempts | Total Marks | Negative Marking |

### Market Share Analysis: Part A's Weightage

In the total addressable market of 200 marks, Part A holds a consistent 15% share. While seemingly small, this segment is often the lowest-hanging fruit, requiring strategic acumen over deep subject specialization, leading to a higher ROI on preparation time.





Topic Frequency & Prioritization
The bar chart below illustrates the distribution of questions across core numerical topics based on an analysis of previous years' papers. It's clear that concepts like Ratios, Percentages, and Averages form the backbone of this section.
25
20
15
10
5
0
Ratio & Proportion
Percentage
Averages
Profit & Loss
Number System
Time & Work
Question Frequency (%)


Competitive Landscape: Logical Reasoning
Logical Reasoning is a diverse segment featuring various problem types. Success here depends on pattern recognition and process mapping. We've visualized both the topic distribution and a typical problem-solving flow to model the required thinking process.
Reasoning Topic Distribution
Series formation and coding-decoding questions are the market leaders in this category, demanding a strong command of alphabetical and numerical patterns.
Series & Analogy
Coding-Decoding
Question Frequency (%)
Value Chain: Direction Sense Problem
Visualizing relationships is key. Below is a simple process flow for a common 'Direction Sense' problem, built entirely with structured HTML and CSS, demonstrating how to map out logical steps without complex tools.
Start at Point O
Walk 4km North
End at Point F (2km North-East of O)

# SECTION 1: Numerical Ability

This section is designed to test your quantitative aptitude. Mastery requires understanding the **Core Concepts**, memorizing key formulas, and practicing a wide variety of problems to improve speed and accuracy.

## 1. Number System

This is the bedrock of quantitative aptitude. A deep understanding of number properties is essential.

### Core Concepts

- **Classification of Numbers:**
  - **Natural Numbers (N):** Counting numbers {1, 2, 3, ...}.
  - **Whole Numbers (W):** Natural numbers including zero {0, 1, 2, 3, ...}.
  - **Integers (Z):** Whole numbers and their negative counterparts {..., -2, -1, 0, 1, 2, ...}.
  - **Rational Numbers (Q):** Numbers that can be expressed as a fraction p/q, where p and q are integers and q=0. Includes terminating decimals (**Example** 0.5 = 1/2) and non-terminating repeating decimals (**Example** 0.333... = 1/3).
  - **Irrational Numbers:** Numbers that cannot be expressed as p/q. They are non-terminating and non-repeating decimals (**Example** 2, π, e).

- Real Numbers (R): The set of all rational and irrational numbers.

- Prime & Composite Numbers:
  - A prime number has exactly two distinct factors: 1 and itself (Example 2, 3, 5, 7, 11). Note: 2 is the only even prime number. 1 is neither prime nor composite.
  - A composite number has more than two factors (Example 4, 6, 8, 9, 10).
- HCF & LCM:
  - HCF (Highest Common Factor) / GCD (Greatest Common Divisor): The largest number that divides two or more numbers without leaving a remainder.
    - Methods: Prime Factorization, Division Method.
  - LCM (Least Common Multiple): The smallest number that is a multiple of two or more numbers.
  - Key Formula: For two numbers A and B, $A \times B = HCF(A,B) \times LCM(A,B)$.
  - For Fractions: HCF of fractions = LCM of DenominatorsHCF of Numerators. LCM of fractions = HCF of DenominatorsLCM of Numerators.[1]

## In-Depth Sub-Topics

- Divisibility Rules:

# SECTION 2: Reasoning

This section evaluates your logical, analytical, and critical thinking skills. It involves pattern recognition, relationship analysis, and drawing valid conclusions.

## 1. Series Formation

Identifying the underlying pattern in a sequence of numbers or letters.

### Core Concepts & Patterns

- **Number Series:**
  - **Arithmetic Series (Constant Difference):** 2, 5, 8, 11, ... (+3)
  - **Geometric Series (Constant Ratio):** 3, 9, 27, 81, ... (x3)
  - **Square/Cube Series:** 1, 4, 9, 16 (n2); 1, 8, 27, 64 (n3); 0, 6, 24, 60 (n3–n).
  - **Two-Stage/Double Difference Series:** The differences between consecutive terms form an AP. Ex: 1, 2, 5, 10, 17... (Differences are 1, 3, 5, 7...).
  - **Mixed Series:** Alternating patterns. Ex: 10, 5, 12, 6, 14, 7... (Pattern 1: 10, 12, 14...; Pattern 2: 5, 6, 7...).
  - **Fibonacci Series:** The next term is the sum of the

previous two. 1, 1, 2, 3, 5, 8...

- **Alphabet Series:** Based on the positional values of letters (A=1, B=2...).

## Illustrative Examples

- **Double Difference Series:** Find the next term: 3, 10, 24, 45, 73, ?
  - **Solution:**
    - Differences (1st level): 7, 14, 21, 28...
    - Differences (2nd level): 7, 7, 7...
    - The next difference in the 1st level will be 28+7=35.
    - The next term in the series will be 73+35=108.
- **Alphabet Series:** Find the missing term: AZ, BY, CX, ?
  - **Solution:** The first letter of each pair moves forward (A, B, C...) and the second letter moves backward (Z, Y, X...). The next pair is **DW**.

## 2. Coding - Decoding

This topic tests your ability to decipher the rule or code used to write a word or message.

### Core Concepts & Types

- **Letter Coding:** Letters are replaced by other letters according to a specific pattern.

# SECTION 3: Data Interpretation & Graphical Analysis

This section applies your numerical skills (averages, percentages, ratios) to analyze data presented in various visual formats. The key is to read carefully and calculate accurately.

## 1. Tabular Data (Tables)

Tables present precise data in a structured row-and-column format.

- **Strategy:**
    1. **Read the Title and Headers:** Understand what the table is about, what each row/column represents, and the units of measurement (**Example** "in thousands," "in million tonnes").
    2. **Scan the Data:** Get a general sense of the values – highs, lows, and trends.
    3. **Focus on the Question:** Pinpoint exactly what data you need. Be careful to pick values from the correct row and column.
    4. **Approximate When Possible:** If options are far apart, you can approximate calculations to save time.

- **Illustrative Example (Advanced Calculation):**

| Year | Company | Sales (in Cr INR) | Profit % |
|---|---|---|---|
| 2022 | A | 500 | 20% |
| 2022 | B | 650 | 15% |
| 2023 | A | 550 | 25% |
| 2023 | B | 600 | 18% |

- **Question:** What was the difference in the profit of Company A and Company B in the year 2023?
    - **Solution:**
        - Profit = Sales × Profit %. (Note: Here Profit % is on Sales, but typically it is on Cost. Assume as per question's context). Let's assume Profit is a percentage of sales.
        - Profit of Company A in 2023 = 25% of 550 = $0.25 \times 550 = 137.5$ Cr.
        - Profit of Company B in 2023 = 18% of 600 = $0.18 \times 600 = 108$ Cr.
        - Difference = $137.5 - 108 = 29.5$ Cr INR.

# SECTION 4: Additional Related Topics

## 1. Clocks

This topic involves problems on the angles between clock hands, correct time, and mirror images.

### Core Concepts

- **Relative Speed:** The minute hand gains 5.5° on the hour hand every minute.
- **Angle Formula:** Angle θ=|30H−211M|.
- **Coincidence (0°), Opposition (180°), Right Angles (90°):**
  - Hands coincide 11 times in 12 hours.
  - Hands are opposite 11 times in 12 hours.
  - Hands are at right angles 22 times in 12 hours.
- **Mirror Image:** To find the mirror image time of a clock, subtract the given time from 11:60 (or 23:60 if the time involves 12).

### Illustrative Examples

- **Faulty Clock:** A clock gains 10 minutes every 2 hours. It was set right at 10 AM on Monday. What will be the correct time when the clock shows 11 AM on Tuesday?
  - **Solution:**
    - The faulty clock gains 5 minutes every hour.

- From 10 AM Monday to 11 AM Tuesday is 25 hours.
- In 1 hour, the faulty clock moves 65 minutes (60 normal + 5 gain).
- When the faulty clock shows 25 hours have passed, the actual time passed = 25×6560=651500≈23.07 hours.
- This is approx. 23 hours and 4 minutes from 10 AM Monday, which is **9:04 AM on Tuesday**.

- **Mirror Image:** What is the mirror image of a clock showing 7:35?
  - **Solution:** Subtract from 11:60.
  - 11:60 - 7:35 = **4:25**.

## 2. Calendars

This topic involves finding the day of the week for a given date based on the concept of "odd days."

### Core Concepts

- **Odd Days:** Number of days more than a complete week.
- **Leap Year:** Divisible by 4 (**Example** 2024). A century year is a leap year only if divisible by 400 (**Example** 2000 is a leap year, 1900 is not).
- **Odd Days Count:**
  - Ordinary Year (365 days) = 1 odd day.
  - Leap Year (366 days) = 2 odd days.

# TABLE OF CONTENT

## UGC NET PAPER-II

## MATHEMATICS

# SEQUENCES OF REAL NUMBERS

# UNIT-I

# SEQUENCES OF REAL NUMBERS

## Introduction (Sequences of Real Numbers)

By a sequence, we mean an arrangement of numbers in a definite order according to some rule. We denote the terms of a sequence by $a_1$, $a_2$, $a_3$, etc., the subscript denotes the position of the term.

In view of the above a sequence in the set X can be regarded as a mapping or a function f : N → X defined by

$$f(n) = a_n \ \forall \ n \in N$$

Domain of f is a set of natural numbers or some subset of it denoting the position of term. If its range denoting the value of terms is a subset of R real numbers then it is called a real sequence. A sequence is either finite or infinite depending upon the number of terms in a sequence. We should not expect that its terms will be necessarily given by a specific formula. However, we expect a theoretical scheme or rule for generating the terms.
Sequence following certain patterns are more often called progressions. In progressions, we note that each term except the first progresses in a definite manner.

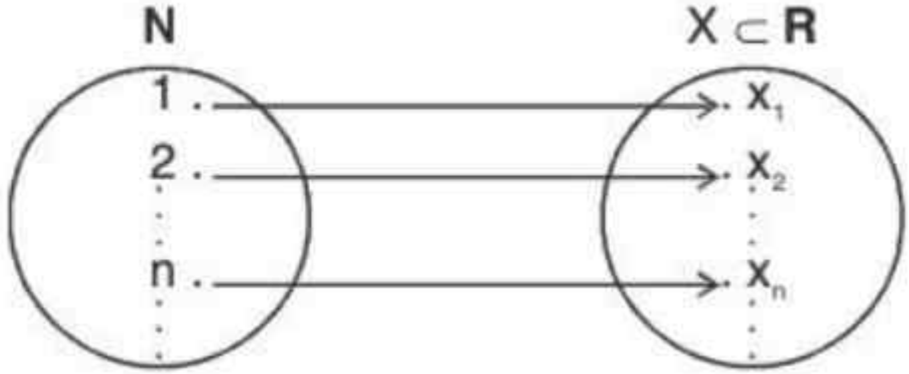
A sequence of real numbers is a function whose domain is a set of the form $\{n \in \mathbb{Z} \mid n \geq m\}$ where m is usually 0 or 1. Thus, a sequence is a function f : N → R. Thus a sequence can be denoted by f(m), f(m + 1), f(m + 2), ... Usually, we will denote such a sequence by $(a_i)^{\infty}_{i=m}$ or $\{a_m, a_{m+1}, a_{m+2}, ....\}$
where $a_i = f(i)$. If m = 1, we may use the notation $\{a_n\}_{n \in N}$.

**Definition:** Every function defined from the set N of natural numbers to a non-empty set X is called a sequence.

## Real Sequence

Every function defined from the set **N** of natural numbers to a non-empty subset X of the set of real numbers **R** is called a real sequence, denoted by f

: N → R.
Thus the real sequence f is set of all ordered pairs {n, f(n)} I {n = 1, 2, 3, ...} i.e., set of all pairs (n, f(n)) with n a positive integer.




If f : **N** → **R** is a sequence, then for each n ∈ **N**, f(n) is a real number. It is conventional to write f(n) as $f_n$.

**Notations:** Since the domain of a sequence is always the same (the set of positive integers) a sequence may be written as {f(n)} instead of {n, f(n)}.

**Example.** The sequence {1, 1/2, 1/3, 1/4, 1/5, ...} is written as $\{1/n\}^{\infty}_{n=1}$. This sequence can be thought of as an ordinary function f(n) = 1/n.

**Example.** Consider the sequence given by $a_n = (-1)^n$ for n ≥ 0. The terms of the sequence look like, { 1, - 1, 1, - 1, 1, - 1, ...}. Note that the function has domain N but the range is { - 1, 1}.

**Example.** Consider the sequence $a_n = \cos\{n\pi/3\}$, n ∈ N. The first terms in the sequence is $\cos\pi/3 = \cos 60^o = 1/2$ and the sequence looks like

$$\{\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, -1, \ldots\}.$$

**Note:-** that the function takes on only a finite number of values, but the sequence has an infinite number of elements.

**Example.** If $a_n = n^{1/n}$, n ∈ N, the sequence is
$1, \sqrt{2}, 3^{1/3}, 4^{1/4}, \ldots.$

**Example.** Consider the sequence $b_n = (1 + 1/n)^n$, $n \in N$. This is the sequence

$$2, \left(\frac{3}{2}\right)^3, \left(\frac{4}{3}\right)^3, \left(\frac{5}{4}\right)^4, \ldots$$

## Representation of A Sequence

The real numbers $x_1, x_2, \ldots, x_n, \ldots$ are called the terms or elements of the sequence. $x_1$, is called the first term, $x_2$ the second term, ... , $x_n$ the nth term of the sequence $\{x_n\}$. It is denoted by $\{x_1, x_2, x_3, \ldots x_n, \ldots\}$ or $\{x_n\}$ or $\{x_n\}$.

**Example.**

$$\left\langle \frac{1}{n} \right\rangle = \left\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\rangle$$
$$\langle 1 + (-1)^n \rangle = \langle 0,2,0,2, \ldots \rangle$$
$$\langle n^2 \rangle = \langle 1,4,9,16, \ldots \rangle$$
$$\langle (-1)^n \rangle = \langle -1,1,-1,1, \ldots \rangle$$

A sequence $(a_n)$ may be defined by a recursion formula:

$$a_{n+1} = \sqrt{2a_n}, a_1 = 1$$
$$a_2 = \sqrt{2a_1} = \sqrt{2 \cdot 1} = \sqrt{2}$$
$$a_3 = \sqrt{2a_2} = \sqrt{2 \cdot \sqrt{2}}$$

Here the terms of the sequence are 1, $\sqrt{2}$,$\sqrt{2\sqrt{2}}$, .....

## Range of a Sequence

The set of all the distinct elements of a sequence is called the range set of the given sequence.

**Example.** The range sets of the sequences given in example are respectively

**(i)** {1, 1/2, 1/3, ...}

**(ii)** {0, 2}

**(iii)** {1, 4, 9, ...}

**(iv)** {-1, 1}

**(v)** $\{1, \sqrt{2}, \sqrt{2\sqrt{2}}, \ldots\}$

**Note:** The range set of a sequence may be finite or infinite but the sequence has always an infinite number of elements.

## Particular Sequences

**(a) Constant Sequence:** If the $n^{th}$ term of the sequence is constant i.e. $a_n = c \in R$, $\forall$ n, then the sequence obtained (c, c, c, ...) is known as constant sequence.

**Example.**

(5, 5, 5, 5, ... ) = (5)

(0, 0, 0, 0, ... ) = (0)

**(b) Identity Sequence:** If the $n^{th}$ term of the sequence is $a_n = n$, then the obtained sequence is called Identity sequence.

**Example.** (1, 2, 3,.....)

## Equal Sequence

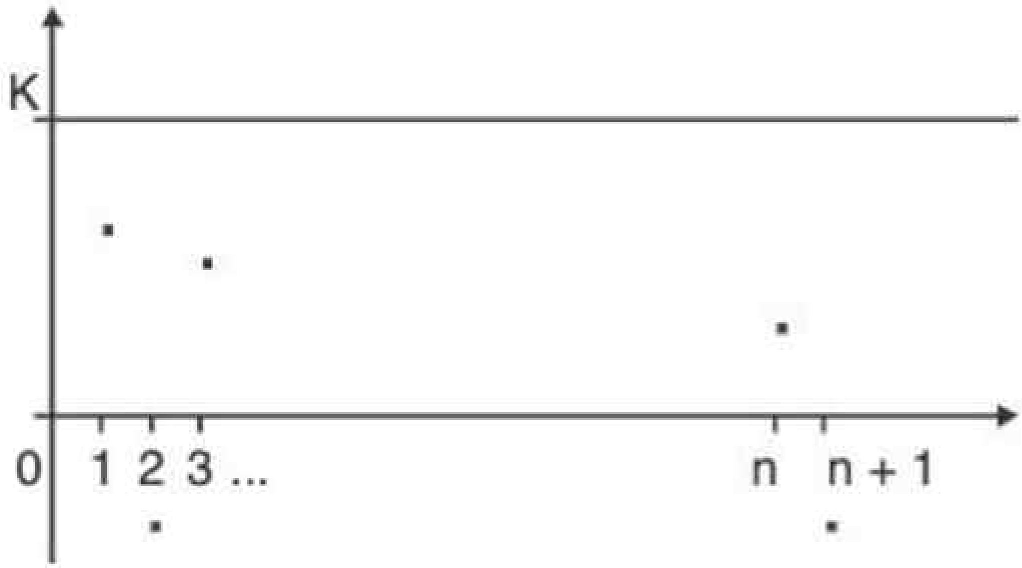
Two sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ are equal, if their $n^{th}$ terms are equal, i.e. $a_n = b_n$, $\forall n \in \mathbf{N}$

## Bounded and Monotone Sequences

### Bounded Sequence

**Bounded above:** A sequence $(a_n)$ is said to be **bound above**, if $\exists$ a real number K s.t.

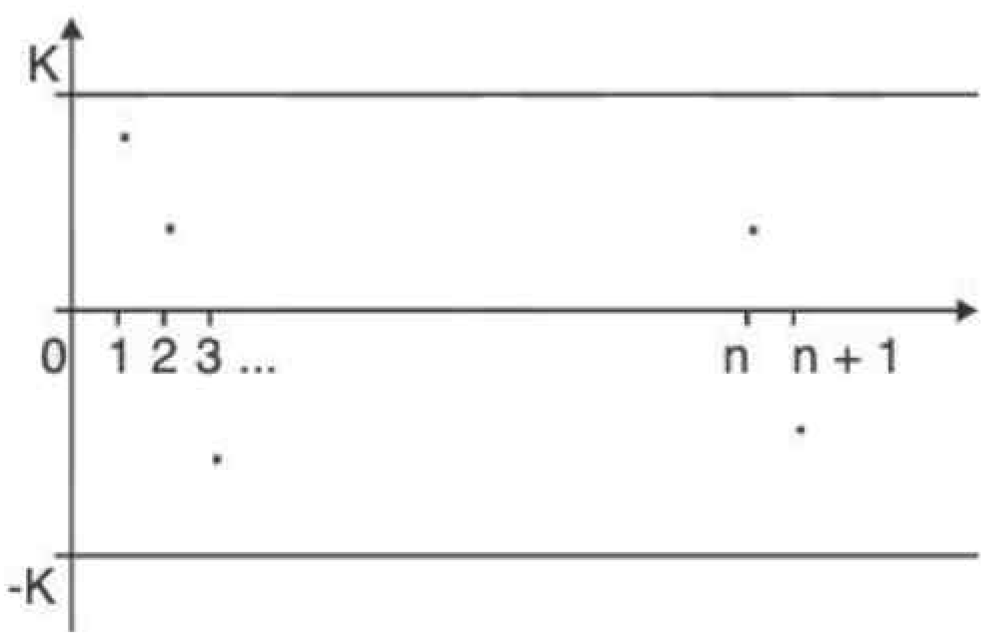
$a_n \leq K \ \forall \ n \in \mathbf{N}$. K is called an **upper bound** of the sequence $(a_n)$.




Bounded above sequence

**Bounded below:** A sequence ($a_n$) is said to be **bounded below**, if ∃ a real number k s.t.
$K \leq a_n \ \forall \ n \in$ **N**. K is called an **lower bound** of the sequence ($a_n$).

**Bounded:** A sequence ($a_n$) is said to be **bounded**, if it is both bounded above and bounded below.
Equivalently, ($a_n$) is **bounded**, if there exist two real numbers k and K such that $k \leq a_n < K \ \forall \ n \in N$.




Bounded sequence

**Examples**

**1.** $(1/n) = (1, 1/2, 1/3, 1/4, ..)$ is bounded, as $0 < 1/n \leq 1\ \forall\ n \in \mathbf{N}$.

**2.** $(1 + (-1)^n) = (0, 2, 0, 2, ...)$ is bounded.

**Supremum and Infimum of a Sequence**

The minimum value of the upper bounds of sequence is known as Supremum or least upper bound (lub) of a sequence.

The greatest of the lower bounds of a sequence is known as the Infimum or greatest lower bound (**gIb**) of a sequence.

**Examples**

- The sequence $\{n^2\}$ is bounded below: $n^2 > 0\ \forall\ n \in \mathbf{N}$ but not bounded above;
- The sequence $\{-n\}$ is bounded above: $-n < 0\ \forall\ n \in N$ but not bounded below;
- The sequence $\{(-1)^n + 1\}$ is bounded : $|(-1)^n + 1| \leq 2\ \forall\ n \in N$.

Infinitely large sequences represent an important subset of unbounded sequences.

**Definition:** A sequence $\{a_n\}$ is called infinitely large if $\forall\ K \in R\ \exists\ n_K \in N$ such that $|a_n| > K\ \forall\ n \geq n_K$.

As an example, we show that the sequence $\{(-1)^n n^3\}$ is infinitely large.

Indeed, for any number K, we can find $n_k$ such that $|(-1)^n n^3| > K\ \forall\ n \geq n_K$.

To this end, we solve the inequality $n^3 > K$, and $n > \sqrt[3]{K}$. t

Let $n_k = [\sqrt[3]{K}] + 1$, where [c] is the integer part of c.

Then for $n \geq n_K$ we obtain.

$n \geq n_K > \sqrt[3]{K} \Rightarrow n^3 > K \Rightarrow |(-1)^n n^3| > K$.

From above Definition it follows that any infinitely large sequence is unbounded. However, the converse is not true: there exist unbounded sequences that are not infinitely large.

For example, such is the sequence $\{(1 - (-1)^n)n\}$.

**Definition.** A sequence $\{a_n\}$ is called infinitely small if

$$\lim_{k\to\infty} a_n = 0,$$

that is for any $\varepsilon > 0$ there exists $n_\varepsilon$ such that

$|a_n| < \varepsilon \ \forall \ n \geq n_\varepsilon$

For example, the sequence $\{q^n\}$ for $|q| < 1$ is infinitely small. Indeed, for any $\varepsilon > 0$ let us find $n\varepsilon$ such that $|q^n| < \varepsilon \ \forall \ n \geq n_\varepsilon$. To this end, we solve the inequality $|q^n| < \varepsilon$, assuming that $0 < \varepsilon < 1$ (for $\varepsilon \geq 1$, this inequality is clearly true for any $n \in N$):

$$\mathbf{n \ln(|q|) < \ln \varepsilon \Rightarrow n > \ln\varepsilon/\ln|q|,\ n_\varepsilon = [\ln\varepsilon/\ln|q|] + 1,}$$

where $\ln\varepsilon < 0$, and $\ln |q| < 0$, since $\varepsilon < 1$, and $|q| < 1$. Thus, for $n \geq n_\varepsilon$ we have

$$\mathbf{n \geq n_\varepsilon > \ln \varepsilon/\ln|q| < \ln\varepsilon \Rightarrow |q|n < \varepsilon.}$$

The fact that $\{a_n\}$ is not infinitely small means the following there exists $\varepsilon_0 > 0$ such that for any $n \in N$ there exists $k_n > n$ with $|a_{k_n}| > \varepsilon_0$.

## Unbounded Sequence

Sequence which is either unbounded above or below is called an unbounded sequence.

## Monotone Sequences

- **Monotonicaliy Increasing:** A sequence $(a_n) = (a_1, a_2, a_3, \ldots, a_n, \ldots)$ is said to be monotonically increasing, if $a_{n+1}, \geq a_n \ \forall \ n \in N$.
- **Monotonically Decreasing:** A sequence $(a_n) = (a_1, a_2, a_3, \ldots. a_n, \ldots>$ is said to be monotonically decreasing, if $a_{n+1}, \leq a_n \ \forall \ n \in N$.
- **Monotonic:** A sequence $(a_n)$ is said to be **monotonic (or monotone)** if it is either monotonically increasing or monotonically decreasing.
- **Strictly monotonic sequence:** either strictly increasing or strictly decreasing. In particular, monotonically increasing is the same as increasing, strictly monotonically increasing the same as strictly increasing.

**Examples**

- The sequence $(2^n)$ = (2, 4, 8, 16, ...) is monotonically increasing.
- Since $a_1 = 2 < a_2 = 4 < a_3 = 8 <$ ......
- The sequence $\left\langle \frac{1}{n} \right\rangle = \left\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\rangle$ is monotonically decreasing.
- Since $a_1 = 1 > a_2 = 1/2 > a_3 = 1/3 >$ .......
- The sequence $\left\{\frac{1}{n^2}\right\} = 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}$ is a bounded monotone decreasing sequence, Its upper bound is greater than or equal to 1, and the lower bound is any non-positive number. The least upper bound is number one, and the greatest lower bound is zero, that is,

$$0 < \frac{1}{n^2} \leq 1$$

- for each natural number n.
- The sequence $\left\{\frac{n}{n+1}\right\} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$ is a bounded monotone increasing sequence. The least upper bound is number one, and the greatest lower bound is 1/2, that is,

$$\frac{1}{2} \leq \frac{n}{n+1} < 1$$

- for each natural number n.
- The sequence $\left\{\frac{n^2}{n+1}\right\} = \frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \ldots$ is an unbounded sequence, because it does not have a finite upper bound.
- Monotone increasing sequences:

  $\{n\} = 1, 2, 3, \ldots$

  $\left\{\frac{n}{n+1}\right\} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$

  $\left\{\frac{n^2}{n+1}\right\} = \frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \ldots$ i

- Monotone decreasing sequences:

  $\left\{\frac{1}{n}\right\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

  $\left\{\frac{1}{n^2}\right\} = 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$

  $\left\{\frac{1}{2^n}\right\} = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

## Eventual nature of a sequence

Definition. A sequence $\{a_n\}$ of real numbers is called non-decreasing if $a_n \leq a_{n+1}$ for all n, and it is called non-increasing if $a_n \geq a_{n+1}$ for all n. It is called strictly increasing if $a_n < a_{n+1}$ for all n, and strictly decreasing if $a_n > a_{n+1}$ for all n.

A sequence $\{a_n\}$ of real numbers is called eventually non-decreasing if there exists a natural number N such that $a_n \leq a_{n+1}$ for all $n \geq N$, and it is called eventually non-increasing if there exists a natural number N such that $a_n \geq a_{n+1}$ for all $n \geq N$. We make analogous definitions of “eventually strictly increasing” and "eventually strictly decreasing.”

## Limit Point of a Sequence

Let $\{a_n\}$ be any sequence and $\alpha \in N$ we say a is limit point of $\{a_n\}$ if every neighbourhood of a contains infinite members of the sequence $\{a_n\}$.

i.e. for any $\delta > 0$

$a_n \in (\alpha - \delta, \alpha + \delta)$ for infinite values of n.

## Examples

**(i)** $a_n = (-1)^n$

$\alpha = -1$

for any $\delta > 0$

$a_n \in (-1 - \delta, -1 + \delta)\ \forall\ n = 2k - 1, k = 1, 2, ....$

$\therefore$ -1 is a limit point.

$\alpha_2 = 1.$

for any $\delta > 0$

$a_n \in (1 - \delta, 1 + \delta)\ \forall\ n = 2k, k = 1, 2, ...$

$\therefore$ 1 is a limit point.

$\{a_n\}$ has two limit points $\{-1, 1\}$.

**(ii)** $a_n = \begin{cases} 2; & n = 1 \text{ or prime.} \\ p; & p|n \text{ and } p \text{ is the least prime doing so.} \end{cases}$

$\alpha = 2$

for any $\delta > 0$

$a_n \in (2 - \delta, 2 + \delta)\ \forall\ n = 1$ or prime

∴ 2 is a limit point.
Let p be any prime,
for any $\delta > 0$
$a_n \in (p - \delta, p + \delta)\ \forall\ n = p^k, = 1, 2, \ldots$
∴ p is the limit point of $a_n$.
Hence, every prime no. is a limit point of $\{a_n\}$.
As set of prime no. are infinite
∴ $\{a_n\}$ has infinite no. of limit points.

**Theorem:** Every limit point of the range set of a sequence is limit point of a sequence.
**Solution:** Let S be range set of sequence $\{a_n\}$.
i.e. S = range of $\{a_n\}$
Let $\alpha \in S'$
for $\varepsilon > 0$
$(\alpha - \varepsilon, \alpha + \varepsilon) \cap S \setminus \{\alpha\}$ has infinite no. of points
Let $q \in (\alpha - \varepsilon, \alpha + \varepsilon) \cap S \setminus \{\alpha\}$
$\Rightarrow q \in (\alpha - \varepsilon, \alpha + \varepsilon)$ and $q \in S$
As $q \in S$
$\Rightarrow q = a_k$ for some $k \in N$
So $a_k \in (\alpha - \varepsilon, \alpha + \varepsilon)$
Hence, $(\alpha - \varepsilon, \alpha + \varepsilon)$ contains infinite no. of terms of sequence
∴ $\alpha$ is limit point of sequence $\{a_n\}$ **Proved**

**Remark:** A real no. is a limit point of sequence ⇔ it appears in the sequence infinite many times or it is the limit point of the range set.

**Bolzano-Weierstrass Theorem**
**Theorem:** Every bounded sequence has a limit point.
**Proof:** Let $(a_n)$ be a bounded sequence.
Let $S = \{a_n : n \in N\}$ be its range.
Since the sequence is bounded, therefore, its range S is also bounded.
**Case I.** Let S be a finite set.
Then there must exist at least one element $\alpha \in S$ such that

$a_n = \alpha$ for an infinite number of values of n.
For any $\varepsilon > 0$, the nbd. $(\alpha - \varepsilon, \alpha + \varepsilon)$ of $\alpha$, contains $a_n = \alpha$, for an infinite number of values of n. Therefore, $\alpha$ is a limit point of $(a_n)$.

**Case II.** Let S be an infinite set.
The range S being an infinite bounded set has a limit point, say p, So each nbd $(p - \varepsilon, p + \varepsilon)$ of p contains an infinite number of elements of S
i.e., $a_n \in (p - \varepsilon, p + \varepsilon)$ for an infinite number of values of n.
Hence p is a limit point of $\langle a_n \rangle$.

## Remark

An unbounded sequence may or may not have a limit point.
Counter example, Since $a_n = n$ is an unbounded sequence with no limit point and $a_n = 1$, if n is even; $a_n = n$, if n is odd is an unbounded sequence with a limit point 1.

## Limit of a Sequence

### Definition

A sequence $\langle a_n \rangle$ is said to have a limit 'l' if for sufficiently large values of n, $| a_n - l |$ can be made as small as we please.
l is the limit of a sequence, if for given $\varepsilon > 0$ $\exists$ $n_o \in N$ s.t.
$| a_n - l | < \varepsilon, \forall n > n_o$
or $\lim a_n = l$

## Some Important Limits

- $\lim\limits_{x \to \infty} (1 + 1/n)^n = e$
- $\lim\limits_{x \to \infty} n^{1/n} = 1,$
- $\lim\limits_{x \to \infty} 1/np = 0$, when $p > 0$ and $p \in R$
- $\lim\limits_{x \to \infty} r^n = 0$, when $| r | < 1$ and $r \in R$
- $\lim\limits_{x \to \infty} r^{1/n} = 1$, when $r > 0$ and $r \in R$

**Example 1: Let $a_n = 1/N$, and let us show that lim $a_n = 0$.**

**Solution:** Given an $\varepsilon > 0$, let us choose a N such that $1/N < \varepsilon$.

Now, if $n \geq N$, then we have

$$|a_n - 0| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon,$$

which is exactly what we needed to show to conclude that lim $a_n = 0$.

**Example 2: Let $a_n = (2n + 1)/(1 - 3n)$, and let $L = -2/3$. Let us show that $L = \lim a_n$.**

**Solution:** Indeed, if $\varepsilon > 0$ is given, we must find a N, such that if $n \geq$ **N** then $|a_n + (2/3)| < \varepsilon$.

Let us examine the quantity $|a_n + 2/3|$. Maybe we can make some estimates on it, in such a way that it becomes clear how to find the natural number **N**.

$$|a_n + (2/3)| = \left|\frac{2n+1}{1-3n} + \frac{2}{3}\right|$$

$$= \left|\frac{6n+3+2-6n}{3-9n}\right|$$

$$= \left|\frac{5}{3-9n}\right|$$

$$= \frac{5}{9n-3}$$

$$= \frac{5}{6n+3n-3}$$

$\leq 5/6n$

$\leq 1/n$,

for all $n \geq 1$. Therefore, if N is an integer for which $N > 1/\varepsilon$, then

$|a_n + 2/3| < 1/n \leq 1/\mathbf{N} < \varepsilon$,

whenever $n \geq N$, as desired.

**Example 3: Let $a_n = 1/\sqrt{N}$, and let us show that lim $a_n = 0$.**

**Solution:** Given an $\varepsilon > 0$, we must find an integer **N** that satisfies the requirements of the definition. It's a little trickier this time to choose this **N**. Consider the positive number $\varepsilon^2$. We know, that there exists a natural number **N** such that $1/N < \varepsilon^2$. Now, if $n \geq N$, then

$$|a_n - 0| = \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} = \sqrt{\frac{1}{N}} < \sqrt{\varepsilon^2} = \varepsilon,$$

which shows that $0 = \lim 1/\sqrt{n}$.

**Theorem:** Prove that a bounded sequence with a unique limit point is convergent.

**Proof:** Let I be the unique limit point of a bounded sequence $\langle a_n \rangle$. Then for any $\varepsilon > 0$.

$a_n \in (I - \varepsilon, I + \varepsilon)$ for infinitely many values of n.

We show that there exists only finitely many values of n, say $m_1, m_2, \ldots, m_k$ such that $a_{m1}, a_{m2}, \ldots a_{mk}$ do not belong to $(I - \varepsilon, I + \varepsilon)$ reason being the infinitely many terms of the given sequence not belonging to $(I - \varepsilon, I + \varepsilon)$ will have a limit point other then I (by **Bolzano-Weierstrass Theorem**), which is a contradiction.

Let $m - 1 = \max\{m_1, m_2, \ldots, m_k\}$. It follows that

$a_n \in (I - \varepsilon, I + \varepsilon)$ for all $n \geq m$

i.e. $|a_n - I| < \varepsilon$ for all $n \geq m$.

Hence $\langle a_n \rangle$ is convergent to I.

## Limit Superior and Limit Inferior of Sequence

Let $\{a_n\}$ be a bounded sequence then the sequence has the least and the greatest limit point.

The least limit point of $\{a_n\}$ is called limit infinite of $\{a_n\}$ and is denoted by $\underset{n \to \infty}{\text{Lt}} \inf a_n$.

**Note:** 1. If $\{a_n\}$ is unbounded above the $\underset{n \to \infty}{\text{Lt}} \sup a_n = \infty$

If $\{a_n\}$ is unbounded below then $\underset{n \to \infty}{\text{Lt}} \inf a_n = -\infty$

2. Since the the greatest limit point of sequence $\{a_n\} \geq$ the least limit point of sequence $\{a_n\}$

$\underset{n \to \infty}{\text{Lt}} \sup a_n \geq \underset{n \to \infty}{\text{Lt}} \inf a_n$.

**Example.**

**(i)** $a_n = (-1)^n$.

Set of limit point = {-1, 1}

$\therefore \underset{n\to\infty}{\text{Lt}} \sup a_n = 1$ and $\underset{n\to\infty}{\text{Lt}} \inf a_n = -1$.

**(ii)** $a_n = 1/n$

Set of limit point = {0}

$\therefore \underset{n\to\infty}{\text{Lt}} \sup a_n = 0 = \underset{n\to\infty}{\text{Lt}} \inf a_n$.

**(iii)** $a_n = (-1)^n$

As sequence is neither bounded above nor bounded below.

$\underset{n\to\infty}{\text{Lt}} \sup a_n = +\infty$ and $\underset{n\to\infty}{\text{Lt}} \inf a_n = -\infty$

**Note:**

If sequence is bounded below then $\underset{n\to\infty}{\text{Lt}} \inf a_n$ = infimum of set of limit point of sequence $a_n$.

If sequence is bounded above then $\underset{n\to\infty}{\text{Lt}} \sup a_n$ = supremum of set of limit point of sequence $a_n$.

If sequence is bounded then $\underset{n\to\infty}{\text{Lt}} \sup a_n$ = supremum of set of limit point of sequence $a_n$.

$\underset{n\to\infty}{\text{Lt}} \inf a_n$ = infimum of set of limit points of sequence $a_n$.

**Classification of Sequence**

1. A sequence is said to be convergent $\Leftrightarrow \underset{n\to\infty}{\text{Lt}} \sup a_n = \underset{n\to\infty}{\text{Lt}} \inf a_n$ (finitely)

2. A sequence is said to be divergent $\Leftrightarrow \underset{n\to\infty}{\text{Lt}} \sup a_n = \underset{n\to\infty}{\text{Lt}} \inf a_n$ (infinitely)

3. A sequence is said to be oscillatory $\Leftrightarrow \underset{n\to\infty}{\text{Lt}} \sup a_n \neq \underset{n\to\infty}{\text{Lt}} \inf a_n$.

**Important**

A sequence of purely irrational number can converge to a rational number And, a sequence of purely rational number can converge to a irrational number.

Let $\alpha \in Q^c$

for any $n \in N$

$$x_n \in \left(\alpha - \frac{1}{n}, \alpha + \frac{1}{n}\right) \cap Q \setminus \alpha$$

then

$$|x_n - \alpha| < \frac{1}{n}$$

Any By Archimedean property; for any $\varepsilon > 0$

$\exists$ m $\in$ N such that

$$\frac{1}{n} < \varepsilon \ \forall\ n \geq m$$

From (1) & (2);

$$|x_n - \alpha| < \frac{1}{n} < \varepsilon \ \forall\ n \geq m$$

i.e. $\quad |x_n - \alpha| < \varepsilon \ \forall\ n \geq m$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ x_n = \alpha\ ; \ \begin{matrix} x_n \neq \alpha \\ x_n \in Q \end{matrix} \ \text{for } n \in N$$

Hence, sequence of purely rational number can converge to irrational number. Similarly, we can prove the other statement.

## Convergent and Divergent Sequence

### Convergent Sequence

**Definition:** A sequence of real numbers is said to converge to a real number L if for every $\varepsilon > 0$ there is an integer $N > 0$ such that if $k > N$ then $|a_k - L| < \varepsilon$. The number L is called the limit of the sequence.

If $\{a_k\}$ converges to L we will write $\lim_{x\to\infty} a_k = L$ or simply $a_k \to L$. If a sequence does not converge, then we say that it **either diverge or oscillate**.

Note that the N in the definition depends on the ε that we were given. If you change the value of ε then you may have to "recalculate" N.
Consider the sequence $a_n = n/2^n$, n ∈ N. Now, if we look at the values that the sequence takes $1/2$, $2/2^2$, $3/2^3$, $4/2^4$, ....

| n | ≈ $n/2^n$ |
|---|---|
| 1 | 0.5 |
| 2 | 0.5 |
| 3 | 0.375 |
| 4 | 0.25 |
| 5 | 0.15625 |
| 6 | 0.09375 |
| 7 | 0.0546875 |
| 8 | 0.03125 |
| 9 | 0.0175781 |
| 10 | 0.00976562 |

we might think that the terms are getting smaller and smaller so maybe the limit of this sequence would be 0. Let's take a look and compare how N would vary as ε varies. Let's start with some simple small numbers and let ε be 0.1, 0.01, 0.001, and 0.0001, and 0.00001. For ε = 0.1, we need to find an integer N so that

$$\left|\frac{N}{2^N} - 0\right| < 0.1$$

Look in the table of values here and we see that for N = 6 we have satisfied the above condition. Following this we get the following by using a calculator or a computer algebra system:

$$\text{N} > 0 \text{ implies } \left|\frac{N}{2^N} - 0\right| < 1$$

$$\text{N} > 5 \text{ implies } \left|\frac{N}{2^N} - 0\right| < 0.1$$

$$N > 9 \text{ implies } \left|\frac{N}{2^N} - 0\right| < 0.01$$

$$N > 14 \text{ implies } \left|\frac{N}{2^N} - 0\right| < 0.001$$
$$N > 18 \text{ implies } \left|\frac{N}{2^N} - 0\right| < 0.0001$$
$$N > 22 \text{ implies } \left|\frac{N}{2^N} - 0\right| < 0.00001$$

A sequence is said to be convergent if its limit is finite.

$\lim_{x \to \infty} Xn = I$

**Definition:** A sequence $<a_n>$ converges to any number I iff for given ε > o ∃$n_0$ ∈ N (depending on ε)
$|a_n - I| < ε \; \forall \; n_o > n$

## Divergent Sequence

A sequence is said to be divergent if $\lim_{x \to \infty} a_n = +\infty$ or $-\infty$

**Definition:** A sequence is said to be divergent if and only if for any k > 0 ∃ a no. $n_o$ s.t.
$a_n > k \; \forall \; n > n_o$.

**For Example,**

**(i)** The sequence $\{a_n\} = \{n\} = \{1, 2, 3, 4, 5, 6, 7, ...\}$ diverges since its limit is infinity (∞).

**(ii)** The sequence $\{a_n\} = \{n^2\} = \{1, 4, 9, 16, 25, 36, 49, ...\}$ diverges

**(iii)** $V_n = (-1)^n$:

This sequence diverges whereas the sequence is bounded:

$-1 \le V_n \le 1$

**(iv)** Another example would be $x_n$ = In n, since

$\ln n \to \infty$ but $\ln n - \ln(n-1) = \ln\frac{n}{n-1} \to \ln 1 = 0.$

**(v)** The sequence $\{\sin(n\pi/2)\}_{n\ge1}$ diverges because the sequence is { 1, 0, - 1. 0, 1, 0, ...}, and hence it does not converge to any number.

## Oscillatory Sequence

The sequence which is neither convergent nor divergent is said to be an oscillatory sequence, i.e., Sequences that tend to nowhere are always oscillating sequences.

**For ex.** (i). $(-1)^n \rightarrow$ nowhere (ii). $(-2)^n \rightarrow$ nowhere

We remark that an unbounded sequence that does not diverge to ∞ or - ∞ oscillates infinity.

For example, the sequences $\{(-1)^n n\}$, $\{(-1)^n n^2\}$, $\{(-n)^n\}$ are all unbounded and oscillate infinitely.

**Theorem:** Every convergent sequence is bounded.

**Proof:** Let us suppose that $<a_n>$ be a convergent sequence and

$\lim_{x \to \infty} a_n = I$.

Let $\varepsilon = 1$. Then there exists a positive integer m such that

$|a_n - I| < 1 \ \forall \ n \geq m$

$\Rightarrow I - 1 < a_n < I + 1 \ \forall \ n \geq m$. (1)

Let $K = \max\{a_1, a_2, \ldots\ldots a_{m-1}, I + 1\}$, (2)

$k = \min\{a_1, a_2, \ldots. a_{m-l}, I - 1\}$. (3)

From (1), (2), (3), we get

$k \leq a \leq K \ \forall \ n$.

Hence $<a_m>$ is bounded.

Converse of this theorem is not true, counter example $a_n = (-1)^n$

**Theorem:** If $\lim_{x \to \infty} a_n = I$, then I is the unique limit point of $<a_n>$

**Proof:** Since $\lim_{x \to \infty} a_n = I$, there exists a positive integer i such that

$|a_n - I| < \varepsilon / 2 \ \forall \ n \geq i$. (1)

Since I' is a limit point of $<a_n>$, there exists a positive integer j such that

$a_n \in (I' - \frac{1}{2}\varepsilon, I' + \frac{1}{2}\varepsilon)$ for n > j (i.e., for infinitely many values of j).

$\Rightarrow |a_n - I'| < \varepsilon / 2$ for n > j. (2)

Let N > max {i, j}

Since N > i, so by (1), $|a_N - l| < \varepsilon / 2$ (3)

And since N > j , so by (2), $|a_N - l'| < \varepsilon / 2$ (4)

Now $|l - l'| = |(a_N - l') - (a_N - l)|$

$\leq |a_N - l'| + |a_N - l| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, by (3) and (4).

Since c is arbitrarily small and $|l - l'| < \varepsilon$, so $|l - l'| = 0$ i.e., $l = l'$.

Hence l is the unique limit point of $\langle a_n \rangle$.

Converse of this theorem also doesn't hold, such as $a_n = 1$, if n is even; $a_n = n$, if n is odd is a sequence with a unique limit point 1 but it does not converge to 1.

**Theorem:** A necessary and sufficient condition for the convergent of a monotonic sequence is that it is bounded.

**Proof:** The condition is necessary (⇒)

Let $\langle a_n \rangle$ be a monotonic and convergent sequence.

We know that every convergent sequence is bounded.

Hence $\langle a_n \rangle$ is bounded.

The condition is sufficient (⇐)

Let $\langle a_n \rangle$ be a monotonic and bounded sequence

Since $\langle a_n \rangle$ is monotonic, we may suppose that $\langle a_n \rangle$ is monotonically increasing. We shall prove that $\langle a_n \rangle$ is convergent.

Let $S = \{a_n : n \in N\}$ be the range of $\langle a_n \rangle$.

Then S is bounded above, since the sequence $\langle a_n \rangle$ is bounded above By order completeness property, S has the least upper bound.

Let p = l.u.b. S. We shall show that $\langle a_n \rangle$ converges to p. Let $\varepsilon > 0$ be any number. Since $p - \varepsilon < p$, $p - \varepsilon$ cannot be an upper bound of S and so there exists some $a_m \in S$ such that $a_m > p - \varepsilon$.

Since $\langle a_n \rangle$ is monotonically increasing, therefore

$a_n \geq a_m \; \forall \; n \geq m$

or $a_n \geq a_m > p - \varepsilon \; \forall \; n \geq m$ or $a_n > p - \varepsilon \; \forall \; n \geq m$ .... (i)

Also $p = \sup S \Rightarrow a_n \leq p < p + \varepsilon \; \forall \; n$. ...(ii)

$\therefore p - \varepsilon < a_n < p + \varepsilon \; \forall \; n \geq m$, using (i), (ii)

or $| a_n - p | < \varepsilon \; \forall \; n \geq m$

Hence $\langle a_n \rangle$ converges to p.

**Theorem: (i)** Every monotonically increasing sequence which is bounded above converge to its least upper bound.

**(ii)** Every monotonically decreasing sequence which is bounded below converge to its greatest lower bound.

Proof, (i) Let $\{a_n\}$ be monotonically increasing sequence which is bounded above.

Let u be the least upper bound of sequence $\{a_n\}$

Let $\varepsilon > 0$

Since $u - \varepsilon < u$

$\therefore$ $u - \varepsilon$ is not an upper bound of $\{a_n\}$

$\Rightarrow \exists\ m \in R$ s. that $a_n > u - \varepsilon$.

Since $a_n$ is monotonically increasing

$a_n \geq a_m > u - \varepsilon \quad \forall\ n \geq m$

$\Rightarrow a_n > u - \varepsilon \qquad \forall\ n \geq m$ ...(1)

Also, u is least upper bound of $\{a_n\}$

$a_n \leq u \qquad\qquad \forall\ n \in N$

$\Rightarrow a_n \leq u + \varepsilon \quad \forall\ n \geq m$ ...(2)

From (1) & (2)

$u - \varepsilon < a_n < u + \varepsilon \quad \forall\ n \geq m$

$\Rightarrow |a_n - u| < \varepsilon \quad \forall\ n \geq m$

$\therefore a_n \to u$.

Similarly, we can prove (2).

**Example 1: Find the value of $\alpha$ for which the sequence <$a_n$> converges to 1 where $a_1 = \alpha$, $a_{n+1} = 1/2\ (a_n^2 + a_n)$ for $N \geq 1$**

**Solution:** <$a_n$> = <1, 1, 1, ... >, if $\alpha = 1$. Hence an $\to 1$, if $\alpha = 1$.

**Example 2: If the sequence ($a_n$), where $a_n = 1 + 1/3 + 1/3^2 + ... + 1/3^{n-1} +$ .... $+ 1/3^{n-1}$ converges then what is the value of $\lim_{x \to \infty} a_n$?**

**Solution:** We have $a_n = 1 + 1/3 + 1/3^2 + .... + 1/3^{n-1}$, which is a geometric progression with common ratio 1/3.

$$\therefore \quad a_n = \frac{\left(1-\frac{1}{3^n}\right)}{\left(1-\frac{1}{3}\right)} = \frac{3}{2}\left(1-\frac{1}{3^n}\right) = \frac{3}{2} - \frac{1}{2}\left(\frac{1}{3}\right)^{n-1}$$

Now,

$$\lim_{n\to\infty} a_n = \frac{3}{2} - \frac{1}{2}\lim_{n\to\infty}\left(\frac{1}{3}\right)^{n-1} = \frac{3}{2} - \frac{1}{2}\times 0 = \frac{3}{2}.$$ [ ∵ lim $r^n = 0$, if I r I < 1].

Hence <$a_n$> is convergent and $\lim_{x\to\infty} a_n = 3/2$.

**Example 3: The sequence ($x_n$) is bounded and monotone where $x_1 = \sqrt{2}$ and $X_{n+1} = \sqrt{2x_n}$, then it converges to 2.**

**Solution:** Observe that $x_2 > x_1$. Since $X^2_{n+1} - X_n^2 = 2(x_n - x_{n-1})$ by induction ($x_n$) is increasing. It can be observed again by induction that $x_n \leq 2$.

Given $x_1 = \sqrt{2}$.

$X_{n+1} = \sqrt{2x_n}$.

Let

$\lim_{n\to\infty} X_n = \ell$

then

$\lim_{n\to\infty} X_{n+1} = \lim_{n\to\infty} \sqrt{2x_n}$

$\Rightarrow \ell = \sqrt{2\ell}$

$\Rightarrow \ell^2 = 2\ell$

$\Rightarrow \ell^2 - 2\ell = 0$

$\Rightarrow \ell(\ell - 2) = o$

$\Rightarrow \ell = 0, 2$

Since $\ell > 0$

$\therefore \ell = 2$

## The Algebra of Convergent Sequences

**Theorem**. If the sequence $\{a_n\}$ converges to L and $c \in R$, then the sequence $\{ca_n\}$ converges to cL; i.e., $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$.

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## CSIR NET Mathematical Sciences Unit 1: Analysis

### Part:- 1 - Real Analysis:

- **Elementary set theory, finite, countable and uncountable sets.**
- **Real number system as a complete ordered field. Archimedean property, supremum, infimum.**
- **Sequences and series of real numbers, convergence, uniform convergence, tests of convergence.**
- **Functions of a single real variable: limits, continuity, differentiability, Mean Value Theorem, Taylor's theorem.**
- **Riemann integral, improper integrals, Riemann-Stieltjes integral.**
- **Functions of several real variables: limits, continuity, directional derivatives, partial derivatives, differentiability, maxima and minima.**
- **Metric spaces, compactness, connectedness, completeness.**
- **Normed linear spaces, Lp spaces**

Q.1 What term describes a set whose elements can be put into one-to-one correspondence with natural numbers?
Answer: Countable set

Q.2 Is the set of rational numbers countable or uncountable?
Answer: Countable

Q.3 Is the set of real numbers countable or uncountable?
Answer: Uncountable

Q.4 What property describes that for any real $x > 0$, there exists a natural number $n$ such that $n > x$?
Answer: Archimedean property

Q.5 What is the least upper bound of a set of real numbers?

Answer: Supremum

Q.6 What is the greatest lower bound of a set of real numbers?
Answer: Infimum

Q.7 What property of real numbers ensures that every non-empty set bounded above has a supremum?
Answer: Completeness property

Q.8 What is the result of the intersection of a countable collection of open sets?
Answer: Gδ set

Q.9 What is the result of the union of a countable collection of closed sets?
Answer: Fσ set

Q.10 What is the cardinality of the power set of natural numbers?
Answer: Continuum

Q.11 What property distinguishes the real number system from the rational number system?
Answer: Completeness

Q.12 Can a bounded sequence in R fail to have a convergent subsequence?
Answer: No

Q.13 What theorem guarantees that every bounded sequence in R has a convergent subsequence?
Answer: Bolzano-Weierstrass Theorem

Q.14 What criterion ensures convergence for sequences in complete metric spaces?
Answer: Cauchy criterion

Q.15 What is the limit of the sequence an=(1+1/n)n?
Answer: e

Q.16 Does the series ∑n=1∞1/n converge or diverge?
Answer: Diverges

Q.17 What is the behavior of the series ∑n=1∞(−1)n/n?
Answer: Converges conditionally

Q.18 What test applies when comparing a series to a known convergent or divergent series?
Answer: Comparison Test

Q.19 What test checks for absolute convergence using the limit of the absolute ratio of successive terms?
Answer: Ratio Test

Q.20 What test checks for absolute convergence using the nth root of the absolute value of the nth term?
Answer: Root Test

Q.21 What test is applicable for alternating series whose terms decrease monotonically to zero?
Answer: Alternating Series Test

Q.22 What is the limit of xsin(1/x) as x→0?
Answer: 0

Q.23 What condition must a function satisfy to be uniformly continuous on an interval?
Answer: δ depends only on ϵ

Q.24 Is f(x)=x2 uniformly continuous on (0,∞)?
Answer: No

Q.25 What theorem states that a continuous function on a closed and bounded interval attains its maximum and minimum values?
Answer: Extreme Value Theorem

Q.26 What theorem states that if f(x) is continuous on [a,b] and differentiable on (a,b), then there exists a c in (a,b) such that f′(c)=(f(b)−f(a))/(b−a)?
Answer: Mean Value Theorem

Q.27 What is the Taylor series expansion of sin(x) around x=0?
Answer: ∑(−1)nx2n+1/(2n+1)!

Q.28 What is the Lagrange form of the remainder in Taylor's Theorem?
Answer: f(n+1)(c)(x−a)n+1/(n+1)!

Q.29 What integral is defined as the limit of Riemann sums?
Answer: Riemann integral

Q.30 What kind of integral extends the concept of definite integrals to unbounded intervals or discontinuous integrands?
Answer: Improper integral

Q.31 What is the convergence criterion for an improper integral of type I (∫a∞f(x)dx)?
Answer: Limit exists

Q.32 What kind of discontinuity in the integrand leads to an improper integral of type II?
Answer: Infinite discontinuity

Q.33 What is a condition for a function to be Riemann integrable on [a,b]?
Answer: Bounded and continuous almost everywhere

Q.34 What theorem gives conditions for a bounded function to be Riemann integrable?
Answer: Lebesgue's Criterion

Q.35 What integral generalizes the Riemann integral by replacing dx with dα(x) where α is a monotonic function?
Answer: Riemann-Stieltjes integral

Q.36 What condition is sufficient for the existence of a Riemann-Stieltjes integral ∫abfdα?
Answer: f continuous, α monotonic

Q.37 If α(x) is differentiable, what is the relation between Riemann-Stieltjes and Riemann integrals?
Answer: ∫abfα′dx

Q.38 What is the main application of the Riemann-Stieltjes integral?
Answer: Probability theory

Q.39 What kind of integral would ∫01(1/x)dx be classified as?

Answer: Improper integral Type II

Q.40 What value does $\int_0^\infty e^{-x}dx$ converge to?
Answer: 1

Q.41 What is the value of $\int_0^1 x^2dx$?
Answer: 1/3

Q.42 What is the Fundamental Theorem of Calculus (Part 1)?
Answer: $\int_a^x f(t)dt=F(x)$ then $F'(x)=f(x)$

Q.43 What is the Fundamental Theorem of Calculus (Part 2)?
Answer: $\int_a^b F'(x)dx=F(b)-F(a)$

Q.44 What is the term for a space where a distance function (metric) is defined between any two points?
Answer: Metric space

Q.45 What are the three axioms that a metric d(x,y) must satisfy?
Answer: Non-negativity, Symmetry, Triangle Inequality

Q.46 What property of a metric space means every Cauchy sequence converges within the space?
Answer: Completeness

Q.47 Is the set of rational numbers Q with the usual metric a complete metric space?
Answer: No

Q.48 Is the set of real numbers R with the usual metric a complete metric space?

Answer: Yes

Q.49 What theorem states that a closed subset of a complete metric space is complete?
Answer: Closed set theorem

Q.50 What property of a metric space means every open cover has a finite subcover?
Answer: Compactness

Q.51 What theorem relates compactness to closedness and boundedness in R^n?
Answer: Heine-Borel Theorem

Q.52 Is every compact subset of a metric space closed and bounded?
Answer: Yes

Q.53 Is every closed and bounded subset of a metric space compact?
Answer: No (only in R^n)

Q.54 What property of a metric space means it cannot be expressed as the union of two disjoint non-empty open sets?
Answer: Connectedness

Q.55 Is the interval (0,1) in R connected?
Answer: Yes

Q.56 Is the set [0,1]∪[2,3] in R connected?
Answer: No

Q.57 What specific property of a metric space implies that points can be approximated by elements from a countable subset?
Answer: Separability

Q.58 What is a metric space where every sequence has a convergent subsequence?
Answer: Sequentially compact

Q.59 What is the relationship between compactness and sequential compactness in metric spaces?
Answer: Equivalent

Q.60 What theorem is a key result for continuous functions on compact sets, guaranteeing uniform continuity?
Answer: Uniform Continuity Theorem

Q.61 What is a vector space equipped with a norm?
Answer: Normed linear space

Q.62 What are the three axioms that a norm ||x|| must satisfy?
Answer: Non-negativity, Homogeneity, Triangle inequality

Q.63 What is a complete normed linear space?
Answer: Banach space

Q.64 What is a Hilbert space?
Answer: Complete inner product space

Q.65 What is the standard Euclidean norm in R^n?
Answer: ||x||2

Q.66 What is the supremum norm for functions, $||f||_\infty$?
Answer: Max absolute value

Q.67 What space consists of functions f such that $\int|f(x)|^p dx<\infty$?
Answer: Lp space

Q.68 For which value of p is Lp space a Hilbert space?
Answer: p=2

Q.69 What inequality relates the integral of a product of two functions in Lp and Lq?
Answer: Holder's inequality

Q.70 What inequality relates the norm of a sum of functions in Lp spaces?
Answer: Minkowski inequality

Q.71 What is the dual space of Lp for $1<p<\infty$?
Answer: Lq where $1/p+1/q=1$

Q.72 What is the dual space of L1?
Answer: $L\infty$

Q.73 What is the dual space of $L\infty$?
Answer: Not L1 (complex)

Q.74 What specific condition must be met for a series of functions to converge uniformly?
Answer: Cauchy criterion for uniform convergence

Q.75 If a sequence of continuous functions converges uniformly, is the limit function continuous?

Answer: Yes

Q.76 If a sequence of differentiable functions converges uniformly, is the limit function differentiable?
Answer: Not necessarily

Q.77 What theorem gives conditions for term-by-term differentiation of a series of functions?
Answer: Term-by-term differentiation theorem

Q.78 What theorem gives conditions for term-by-term integration of a series of functions?
Answer: Term-by-term integration theorem

Q.79 What is the definition of a limit of a function f(x) as $x \to a$?
Answer: $\epsilon$-$\delta$ definition

Q.80 What type of discontinuity does $f(x)=\sin(1/x)$ have at $x=0$?
Answer: Essential discontinuity

Q.81 What type of discontinuity does $f(x)=x/|x|$ have at $x=0$?
Answer: Jump discontinuity

Q.82 What does differentiability of a function imply about its continuity?
Answer: Implies continuity

Q.83 What does continuity of a function imply about its differentiability?
Answer: Not necessarily

Q.84 What condition must a function satisfy for Rolle's Theorem to apply?
Answer: $f(a)=f(b)$

Q.85 What is the extension of Mean Value Theorem to two functions?
Answer: Cauchy's Mean Value Theorem

Q.86 What is the integral of $f(x)=\chi Q(x)$ (indicator function of rationals) on [0,1]?
Answer: Not Riemann integrable

Q.87 What kind of integrals are used to define Fourier Transforms?
Answer: Improper integrals

Q.88 What is a function of several real variables $f:Rn \to R$?
Answer: Multivariable function

Q.89 What is the definition of a limit for a function of several variables?
Answer: $\epsilon$-$\delta$ definition (multivariable)

Q.90 What type of derivative is computed with respect to one variable, holding others constant?
Answer: Partial derivative

Q.91 What type of derivative is computed along a specific vector direction?
Answer: Directional derivative

Q.92 What theorem states conditions for equality of mixed partial derivatives (e.g., fxy=fyx)?
Answer: Clairaut's Theorem (Schwarz's)

Q.93 What test uses the Hessian matrix to classify critical points (maxima, minima, saddle points)?
Answer: Second derivative test

Q.94 What is a point where the gradient of a function is zero or undefined?
Answer: Critical point

Q.95 What theorem deals with the existence of implicit functions defined by equations?
Answer: Implicit Function Theorem

Q.96 What theorem deals with the existence of inverse functions for differentiable maps?
Answer: Inverse Function Theorem

Q.97 What is a condition for a function to be differentiable at a point in multiple dimensions?
Answer: Linear approximation exists

Q.98 What is the definition of a Cauchy sequence in a metric space?
Answer: Terms get arbitrarily close

Q.99 What is the completion of a metric space?
Answer: Smallest complete space containing it

Q.100 What is the completion of the rational numbers Q?
Answer: Real numbers R


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## CSIR NET Mathematical Sciences Unit 1: Analysis

Part:- 1 - Real Analysis:

- Elementary set theory, finite, countable and uncountable sets.
- Real number system as a complete ordered field. Archimedean property, supremum, infimum.
- Sequences and series of real numbers, convergence, uniform convergence, tests of convergence.
- Functions of a single real variable: limits, continuity, differentiability, Mean Value Theorem, Taylor's theorem.
- Riemann integral, improper integrals, Riemann-Stieltjes integral.
- Functions of several real variables: limits, continuity, directional derivatives, partial derivatives, differentiability, maxima and minima.
- Metric spaces, compactness, connectedness, completeness.
- Normed linear spaces, Lp spaces.

  (100 MCQs)

Q1. Which of the following sets is uncountable?

A) The set of natural numbers (N).

B) The set of integers (Z).

C) The set of rational numbers (Q).

D) The set of real numbers (R).

Answer: D) The set of real numbers (R).

Q2. The "Archimedean property" of real numbers states that for any real number x, there exists a natural number n such that:

A) $n < x$

B) $n = x$

C) $n > x$

D) $n \geq x$

Answer: C) $n > x$

Q3. A sequence of real numbers {a_n} is convergent if and only if it is:
A) Monotonic.
B) Bounded.
C) A Cauchy sequence.
D) Divergent.
Answer: C) A Cauchy sequence.

Q4. Assertion (A): If a function f is continuous on a closed and bounded interval [a, b], then it is uniformly continuous on [a, b].
Reason (R): This is a direct consequence of the Heine-Borel theorem applied to the interval [a, b].
A) Both A and R are true and R is the correct explanation of A.
B) Both A and R are true but R is NOT the correct explanation of A.
C) A is true but R is false.
D) A is false but R is true.
Answer: A) Both A and R are true and R is the correct explanation of A.

Q5. The series Σ(1/n^p) converges if:
A) $p < 1$
B) $p = 1$
C) $p > 1$
D) $p \leq 1$
Answer: C) $p > 1$ (p-series test)

Q6. The "supremum" of a non-empty set S of real numbers that is bounded above is:
A) The largest element in S.
B) The smallest element in S.
C) The least upper bound of S.
D) The greatest lower bound of S.
Answer: C) The least upper bound of S.

Q7. Statement I: If a function is differentiable at a point, then it is continuous at that point.
Statement II: If a function is continuous at a point, then it is differentiable at that point.
A) Both Statement I and Statement II are true.
B) Both Statement I and Statement II are false.
C) Statement I is true, but Statement II is false.
D) Statement I is false, but Statement II is true.
Answer: C) Statement I is true, but Statement II is false. (e.g., |x| at x=0 is continuous but not differentiable)

Q8. According to the Mean Value Theorem, if f is continuous on [a, b] and differentiable on (a, b), then there exists a c in (a, b) such that:
A) f'(c) = 0
B) f'(c) = f(b) - f(a)
C) f'(c) = (f(b) - f(a)) / (b - a)
D) f(c) = (f(a) + f(b)) / 2
Answer: C) f'(c) = (f(b) - f(a)) / (b - a)

Q9. Match List I with List II regarding properties of sets of real numbers:
List I
a. Finite set
b. Countably infinite set
c. Uncountable set
d. Empty set
List II

1. Can be put into one-to-one correspondence with the set of natural numbers.
2. Contains no elements.
3. Contains a finite number of elements.

4. Cannot be put into one-to-one correspondence with the set of natural numbers.

Which of the following is the correct match?

A) a-3, b-1, c-4, d-2

B) a-1, b-2, c-3, d-4

C) a-2, b-3, c-4, d-1

D) a-4, b-1, c-2, d-3

Answer: A) a-3, b-1, c-4, d-2

Q10. The Riemann integral of a function f over an interval [a, b] exists if f is:

A) Continuous on [a, b].

B) Bounded on [a, b].

C) Monotonic on [a, b].

D) Both A and B.

Answer: A) Continuous on [a, b]. (Continuity implies integrability)

Q11. Assertion (A): The set of rational numbers (Q) is dense in the set of real numbers (R).

Reason (R): Between any two distinct real numbers, there exists infinitely many rational numbers.

A) Both A and R are true and R is the correct explanation of A.

B) Both A and R are true but R is NOT the correct explanation of A.

C) A is true but R is false.

D) A is false but R is true.

Answer: A) Both A and R are true and R is the correct explanation of A.

Q12. A function f: (a, b) → R is uniformly continuous if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x, y in (a, b), if $|x - y| < \delta$, then:

A) $|f(x) - f(y)| < \varepsilon$

B) $|f(x) - f(y)| > \varepsilon$

C) $|f(x) - f(y)| = \varepsilon$

D) $f(x) = f(y)$

Answer: A) $|f(x) - f(y)| < \varepsilon$

Q13. Statement I: Every convergent sequence is bounded.

Statement II: Every bounded sequence is convergent.

A) Both Statement I and Statement II are true.

B) Both Statement I and Statement II are false.

C) Statement I is true, but Statement II is false.

D) Statement I is false, but Statement II is true.

Answer: C) Statement I is true, but Statement II is false. (e.g., (-1)^n is bounded but not convergent)

Q14. For a function of several real variables f(x, y), the directional derivative in the direction of a unit vector u is:

A) The partial derivative with respect to x.

B) The partial derivative with respect to y.

C) $\nabla f \cdot u$

D) $|\nabla f|$

Answer: C) $\nabla f \cdot u$

Q15. Which of the following tests can be used to determine the convergence of a series Σa_n if lim |a_n|^(1/n) < 1?

A) Ratio test

B) Root test

C) Comparison test

D) Integral test

Answer: B) Root test

Q16. Assertion (A): The interval (0, 1) is an open set in R.

Reason (R): For every point x in (0, 1), there exists an ε-neighborhood (x - ε, x + ε) that is entirely contained within (0, 1).

A) Both A and R are true and R is the correct explanation of A.

B) Both A and R are true but R is NOT the correct explanation of A.

C) A is true but R is false.

D) A is false but R is true.

Answer: A) Both A and R are true and R is the correct explanation of A.

Q17. Match List I with List II regarding properties of functions:

List I

a. Continuous function

b. Differentiable function

c. Riemann integrable function

d. Uniformly continuous function

List II

1. Implies continuity.
2. Implies boundedness.
3. Implies uniform continuity on closed and bounded intervals.
4. Requires specific ε-δ definition independent of point.

Which of the following is the correct match?

A) a-3, b-1, c-2, d-4

B) a-1, b-2, c-3, d-4

C) a-2, b-3, c-4, d-1

D) a-4, b-1, c-2, d-3

Answer: A) a-3, b-1, c-2, d-4

Q18. The "infimum" of a non-empty set S of real numbers that is bounded below is:

A) The smallest element in S.

B) The largest element in S.

C) The greatest lower bound of S.

D) The least upper bound of S.
Answer: C) The greatest lower bound of S.

Q19. The sum of the series Σ(1/n) (harmonic series) is:
A) 0
B) 1
C) Converges
D) Diverges
Answer: D) Diverges

Q20. A function f(x) is said to have a local maximum at c if there exists an open interval I containing c such that f(x) ≤ f(c) for all x in I.
A) True
B) False
Answer: A) True

Q21. A sequence {a_n} converges to L if for every ε > 0, there exists a natural number N such that for all n > N, |a_n - L| < ε. This is the definition of:
A) Boundedness.
B) Monotonicity.
C) Convergence.
D) Cauchy sequence.
Answer: C) Convergence.

Q22. Assertion (A): The set of integers (Z) is a countably infinite set.
Reason (R): We can put the elements of Z into a one-to-one correspondence with the elements of the natural numbers (N) (e.g., 0, 1, -1, 2, -2, ...).
A) Both A and R are true and R is the correct explanation of A.
B) Both A and R are true but R is NOT the correct explanation of A.

C) A is true but R is false.
D) A is false but R is true.
Answer: A) Both A and R are true and R is the correct explanation of A.

Q23. Statement I: If f is continuous on [a, b], then f attains its maximum and minimum values on [a, b]. (Extreme Value Theorem)
Statement II: If f is continuous on (a, b), then f attains its maximum and minimum values on (a, b).
A) Both Statement I and Statement II are true.
B) Both Statement I and Statement II are false.
C) Statement I is true, but Statement II is false.
D) Statement I is false, but Statement II is true.
Answer: C) Statement I is true, but Statement II is false. (e.g., f(x) = x on (0, 1) doesn't attain max/min)

Q24. For a function f(x, y), if its partial derivatives ∂f/∂x and ∂f/∂y exist at a point, then the function is:
A) Always differentiable at that point.
B) Always continuous at that point.
C) Not necessarily differentiable or continuous at that point.
D) Always has a local extremum.
Answer: C) Not necessarily differentiable or continuous at that point.

Q25. The "Weierstrass M-test" is used to determine the:
A) Convergence of a sequence.
B) Uniform convergence of a series of functions.
C) Differentiability of a function.
D) Integrability of a function.
Answer: B) Uniform convergence of a series of functions.

Q26. Assertion (A): The set of all algebraic numbers is countable.

Reason (R): An algebraic number is a root of a non-zero polynomial with integer coefficients, and there are only countably many such polynomials.

A) Both A and R are true and R is the correct explanation of A.

B) Both A and R are true but R is NOT the correct explanation of A.

C) A is true but R is false.

D) A is false but R is true.

Answer: A) Both A and R are true and R is the correct explanation of A.

Q27. Match List I with List II regarding types of integrals:

List I

a. Riemann integral

b. Improper integral of Type I

c. Improper integral of Type II

d. Riemann-Stieltjes integral

List II

1. Integral over an infinite interval.
2. Generalization of Riemann integral with respect to a function of bounded variation.
3. Integral where the integrand has a discontinuity within the interval.
4. Defined using Riemann sums for continuous functions over a finite interval.

Which of the following is the correct match?

A) a-4, b-1, c-3, d-2

B) a-1, b-2, c-3, d-4

C) a-2, b-3, c-4, d-1

D) a-3, b-4, c-1, d-2

Answer: A) a-4, b-1, c-3, d-2

Q28. A "metric space" (X, d) is a set X with a metric d, which defines a:

A) Linear ordering.

B) Distance between any two elements in X.

C) Algebraic operation.
D) Topological space.
Answer: B) Distance between any two elements in X.

Q29. Statement I: Every compact set in a metric space is closed and bounded.
Statement II: Every closed and bounded set in $R^n$ is compact. (Heine-Borel Theorem)
A) Both Statement I and Statement II are true.
B) Both Statement I and Statement II are false.
C) Statement I is true, but Statement II is false.
D) Statement I is false, but Statement II is true.
Answer: A) Both Statement I and Statement II are true.

Q30. The "supremum" of the set S = {1 - 1/n : n ∈ N} is:
A) 0
B) 1/2
C) 1
D) ∞
Answer: C) 1

Q31. Statement I: A sequence {a_n} converges to L if and only if every subsequence of {a_n} converges to L.
Statement II: If a sequence {a_n} is monotonic and bounded, then it is convergent.
A) Both Statement I and Statement II are true.
B) Both Statement I and Statement II are false.
C) Statement I is true, but Statement II is false.
D) Statement I is false, but Statement II is true.
Answer: A) Both Statement I and Statement II are true.

Q32. Assertion (A): The set of all real numbers (R) is a complete ordered field.
Reason (R): It has the least upper bound property, meaning every non-empty set of real numbers that is bounded above has a supremum in R.
A) Both A and R are true and R is the correct explanation of A.
B) Both A and R are true but R is NOT the correct explanation of A.
C) A is true but R is false.
D) A is false but R is true.
Answer: A) Both A and R are true and R is the correct explanation of A.

Q33. The condition for a function f(x, y) to be differentiable at a point (a, b) is that its linear approximation accurately represents the function's change. This implies that:
A) Its partial derivatives exist.
B) Its partial derivatives are continuous in a neighborhood of (a, b).
C) It is continuous at (a, b).
D) All of the above (implicitly related to differentiability criteria).
Answer: B) Its partial derivatives are continuous in a neighborhood of (a, b). (This is a sufficient condition for differentiability. Existence of partials is not sufficient.)

Q34. The "Taylor series" expansion of a function f(x) around x=a is given by:
A) Σ f^(n)(a) / n! * $(x - a)^n$
B) Σ f^(n)(x) / n! * $(x - a)^n$
C) f(a) + f'(a)(x-a)
D) Σ $(x - a)^n$
Answer: A) Σ f^(n)(a) / n! * $(x - a)^n$

Q35. Match List I with List II regarding convergence tests for series:
List I

a. Ratio Test
b. Root Test
c. Comparison Test
d. Integral Test
List II
1. Compares a series with a known convergent or divergent series.
2. Applies if lim |a_n+1 / a_n| < 1.
3. Applies if lim |a_n|^(1/n) < 1.
4. Compares a series to an improper integral.

Which of the following is the correct match?
A) a-2, b-3, c-1, d-4
B) a-1, b-2, c-3, d-4
C) a-3, b-4, c-1, d-2
D) a-4, b-1, c-2, d-3
Answer: A) a-2, b-3, c-1, d-4

Q36. The "interior point" of a set S in a metric space is a point x such that:
A) x is in S.
B) Every neighborhood of x contains points not in S.
C) There exists an ε-neighborhood of x entirely contained within S.
D) x is a limit point of S.
Answer: C) There exists an ε-neighborhood of x entirely contained within S.

Q37. Assertion (A): Every bounded sequence of real numbers has a convergent subsequence. (Bolzano-Weierstrass Theorem)
Reason (R): This theorem is fundamental in real analysis for proving existence of limits and convergence.
A) Both A and R are true and R is the correct explanation of A.
B) Both A and R are true but R is NOT the correct explanation of A.
C) A is true but R is false.
D) A is false but R is true.

Answer: A) Both A and R are true and R is the correct explanation of A.

Q38. Statement I: A sequence {a_n} is bounded if there exists a real number M such that |a_n| ≤ M for all n.
Statement II: A monotonic sequence is either always increasing or always decreasing.
A) Both Statement I and Statement II are true.
B) Both Statement I and Statement II are false.
C) Statement I is true, but Statement II is false.
D) Statement I is false, but Statement II is true.
Answer: A) Both Statement I and Statement II are true.

Q39. For a real-valued function f(x) defined on an interval [a, b], the "lower Riemann sum" is always:
A) Greater than or equal to the upper Riemann sum.
B) Less than or equal to the upper Riemann sum.
C) Equal to the upper Riemann sum for integrable functions.
D) Equal to zero.
Answer: B) Less than or equal to the upper Riemann sum.

Q40. Match List I with List II regarding types of sets in real analysis:
List I
a. Open set
b. Closed set
c. Compact set
d. Connected set
List II

1. Contains all its limit points.
2. Cannot be expressed as a union of two disjoint non-empty open sets.
3. Every point is an interior point.
4. Every open cover has a finite subcover.

Which of the following is the correct match?
A) a-3, b-1, c-4, d-2
B) a-1, b-2, c-3, d-4
C) a-2, b-3, c-4, d-1
D) a-4, b-1, c-2, d-3
Answer: A) a-3, b-1, c-4, d-2

Q41. The "Lebesgue integral" is a generalization of the Riemann integral that allows for integration of:
A) Only continuous functions.
B) A wider class of functions, including those with many discontinuities.
C) Only monotonic functions.
D) Only polynomials.
Answer: B) A wider class of functions, including those with many discontinuities.
(Y42.** Assertion (A): A function f: [a, b] → R is Riemann integrable if and only if its set of discontinuities has measure zero.
Reason (R): The set of discontinuities of a Riemann integrable function on a closed interval must be of measure zero.
A) Both A and R are true and R is the correct explanation of A.
B) Both A and R are true but R is NOT the correct explanation of A.
C) A is true but R is false.
D) A is false but R is true.
Answer: A) Both A and R are true and R is the correct explanation of A.

Q43. Statement I: The domain of a function of several real variables is a subset of $R^n$.
Statement II: For a function f(x, y), its partial derivative with respect to x (∂f/∂x) is calculated by treating y as a constant.
A) Both Statement I and Statement II are true.
B) Both Statement I and Statement II are false.

C) Statement I is true, but Statement II is false.
D) Statement I is false, but Statement II is true.
Answer: A) Both Statement I and Statement II are true.

Q44. The "Cauchy criterion for convergence of series" states that the series $\Sigma a_n$ converges if and only if for every $\varepsilon > 0$, there exists N such that for all $m > n \geq N$, $|a_n + a_{n+1} + \ldots + a_m| < \varepsilon$.
A) True
B) False
Answer: A) True

Q45. Which of the following defines a "norm" on a vector space V?
A) A function $||.|| : V \to R$ such that $||v|| \geq 0$, $||v|| = 0$ iff $v=0$, $||cv|| = c||v||$, and $||v+w|| \leq ||v|| + ||w||$.
B) A function $||.|| : V \to R$ such that $||v|| \geq 0$, $||v|| = 0$ iff $v=0$, $||cv|| = |c| \, ||v||$, and $||v+w|| \leq ||v|| + ||w||$.
C) A function $||.|| : V \to R$ such that $||v|| = 0$.
D) A function $||.|| : V \to R$ such that $||v+w|| = ||v|| + ||w||$.
Answer: B) A function $||.|| : V \to R$ such that $||v|| \geq 0$, $||v|| = 0$ iff $v=0$, $||cv|| = |c| \, ||v||$, and $||v+w|| \leq ||v|| + ||w||$.

Q46. Assertion (A): The derivative of a function at a point represents the slope of the tangent line to the function's graph at that point.
Reason (R): It is defined as the limit of the difference quotient as the increment approaches zero.
A) Both A and R are true and R is the correct explanation of A.
B) Both A and R are true but R is NOT the correct explanation of A.
C) A is true but R is false.
D) A is false but R is true.
Answer: A) Both A and R are true and R is the correct explanation of A.

Q47. Match List I with List II regarding properties of functions of several real variables:

List I

a. Partial derivative

b. Directional derivative

c. Differentiable

d. Local maximum

List II

1. Rate of change of function along a specific direction.
2. All partial derivatives exist and are continuous.
3. Function value is greater than or equal to values in a neighborhood.
4. Rate of change of function with respect to one variable, holding others constant.

Which of the following is the correct match?

A) a-4, b-1, c-2, d-3

B) a-1, b-2, c-3, d-4

C) a-2, b-3, c-4, d-1

D) a-3, b-4, c-1, d-2

Answer: A) a-4, b-1, c-2, d-3

Q48. The "Lipschitz condition" is a stronger condition than uniform continuity. It states that:

A) The function is continuous.

B) There exists a constant K such that $|f(x) - f(y)| \leq K|x - y|$ for all x, y in the domain.

C) The function is differentiable.

D) The function is monotonic.

Answer: B) There exists a constant K such that $|f(x) - f(y)| \leq K|x - y|$ for all x, y in the domain.

Q49. Statement I: Every convergent sequence in a metric space is a Cauchy sequence.

Statement II: Every Cauchy sequence in a metric space is a convergent sequence.

A) Both Statement I and Statement II are true.

B) Both Statement I and Statement II are false.

C) Statement I is true, but Statement II is false.

D) Statement I is false, but Statement II is true.

Answer: C) Statement I is true, but Statement II is false. (Only if the metric space is complete, then Cauchy implies convergent.)

Q50. Match List I with List II regarding types of sets in set theory:

List I

a. Countable set

b. Uncountable set

c. Denumerable set

d. Finite set

List II

1. A set that can be put into one-to-one correspondence with the natural numbers.
2. A set that contains a finite number of elements.
3. A set that is either finite or countably infinite.
4. A set that is not countable.

Which of the following is the correct match?

A) a-3, b-4, c-1, d-2

B) a-1, b-2, c-3, d-4

C) a-2, b-3, c-4, d-1

D) a-4, b-1, c-2, d-3

Answer: A) a-3, b-4, c-1, d-2